STEADY-STATE TEMPERATURE FIELD OF A FINITE CYLINDER IN A HOUSING

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An analytic solution has been obtained for the steady-state temperature field of two coaxial finite cylinders, the inner of which is a rotor. Two cases corresponding to different conditions of cooling of the outer cylinder, or housing, are considered.

In solving problems involving determination of the temperature fields of gyro motors, miniature control motors, and certain types of electric drive, it is necessary to consider the temperature field of two coaxial cylinders of finite length. The inner cylinder plays the part of rotor, while the outer cylinder represents the housing or case, as shown schematically in the figure.

The heat sources are the stator, located in the interior cavity of the rotor, and the rotor bearings. Heat escapes through the housing to the external medium.

We will consider the problem of determining the steady-state temperature field for a model with the properties described above and boundary conditions of practical importance.

The following quantities are assumed known: $T_{st}(z)$ the temperature distribution along the surface of the stator, Θ the temperature of the medium surrounding the housing, and W_1, W_2 the heat outputs of the right and left bearings. We further assume that the temperature does not vary over the thickness of the housing.

In [1] the equation of heat conduction is derived for a thin plate both of whose surfaces are cooled.

This equation is easily adapted to the case in which one surface of the plate is heated by a gas flow and the other is cooled. Moreover, after going over to a cylindrical coordinate system, the equation thus obtained can be applied to a thin-walled cylindrical shell. With the above assumptions the determination of the temperature field of the rotor and the housing reduces to the solution of the system of equations

$$\frac{\partial^2 T_r}{\partial r^2}+\frac{1}{r}\frac{\partial T_r}{\partial r}+\frac{\partial^2 T_r}{\partial z^2}=0,$$
$$\frac{d^2 T_{0c}}{dz^2}\pm\gamma_1^2(T_{0c}-\Theta)=0,$$
$$\frac{d^2 T_{ic}}{dr^2}+\frac{1}{r}\frac{dT_{ic}}{dr}\pm\gamma_2^2(T_{ic}-\Theta)=0 \quad (i=1,\ 2) \tag{1}$$

with boundary conditions

$$\left[T_r-\frac{1}{\beta_b}\frac{\partial T_r}{\partial r}\right]_{r=r_0}=T_{st},$$
$$\left[T_r-\frac{1}{\beta_0}\frac{\partial T_r}{\partial r}\right]_{r=R_0}=\left[T_{0c}-\frac{1}{\beta_0}\frac{\partial T_{0c}}{\partial r}\right],$$
$$\left[T_r-\frac{1}{\beta_1}\frac{\partial T_r}{\partial z}\right]_{z=l}=\left[T_{1c}-\frac{1}{\beta_2}\frac{\partial T_{1c}}{\partial z}\right],$$
$$\left[T_r-\frac{1}{\beta_1}\frac{\partial T_r}{\partial z}\right]_{z=-l}=\left[T_{2c}-\frac{1}{\beta_2}\frac{\partial T_{2c}}{\partial z}\right]. \tag{2}$$

The quantities γ_1^2 and γ_2^2 are found from the equations

$$\gamma_1^2=\frac{\beta_0}{\delta(\delta\beta_0-1)}-\frac{\beta_3}{\delta};\quad \gamma_2^2=\frac{\beta_2}{\delta(\delta\beta_2-1)}-\frac{\beta_3}{\delta}. \tag{3}$$

As may be seen from expressions (3), for different relations between β_i and δ the quantities γ_1^2 and γ_2^2 may be both positive and negative.

The substitution of

$$T_r(r,\ z)=R(r)Z(z) \tag{4}$$

in the first equation of system (1) leads to the equation

$$\frac{1}{Z}\frac{d^2 Z}{dz^2}=-\frac{1}{R}\left(\frac{d^2 R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)=-m^2. \tag{5}$$

We take the eigenfunctions $T_k(r, z)$ in the form

$$T_k=[B_{1k}I_0(m_k r)+B_{2k}K_0(m_k r)]\cos(m_k z)+$$
$$+[B_{3k}I_0(m_k r)+B_{4k}K_0(m_k r)]\sin(m_k z). \tag{6}$$

In order to satisfy boundary conditions (2) for an arbitrary form of the functions $T_{st}(z)$, $T_{0c}(z)$, $T_{ic}(z)$, we represent the function T, in accordance with [2], in the form of a series in eigenfunctions:

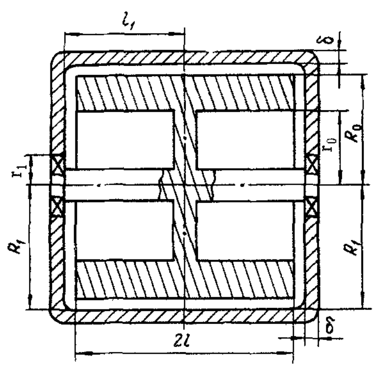
$$T=T_0(r)+\sum_{k=1}^{\infty}(T_k'+T_k''). \tag{7}$$

Here, $T_0(r)$ is the solution of Eqs. (5) at $m = 0$; T_k' and T_k'' are functions determined by expression (6) and satisfying, instead of (2), the following boundary conditions:

$$T_k'|_{z=\pm l}=0;$$
$$\left[\frac{\partial T_k'}{\partial r}+\beta_b T_k'\right]_{r=r_0}=F_{0k}(z);$$
$$\left[\frac{\partial T_k'}{\partial r}+\beta_0 T_k'\right]_{r=R_0}=F_{1k}(z);$$
$$T_k''|_{r=r_0}=0;\quad T_k''|_{r=R_0}=0;$$
$$\left[\frac{\partial T_k''}{\partial z}+\beta_1 T_k''\right]_{z=l}=F_{2k}(r);$$
$$\left[\frac{\partial T_k''}{\partial z}+\beta_1 T_k''\right]_{z=-l}=F_{3k}(r). \tag{8}$$

The functions F_{ik} in expressions (8) are the k-th harmonics of the Fourier expansions in eigenvalues of the following functions:

$$F_0=\beta_b T_{st};\quad F_1=\beta_0 T_{0c}-\frac{\partial T_{0c}}{\partial r};$$
$$F_2=\beta_2 T_{1c}-\frac{\partial T_{1c}}{\partial z};\quad F_3=\beta_2 T_{2c}-\frac{\partial T_{2c}}{\partial z}. \tag{9}$$



Schematic representation of finite cylinder in a housing.

From the boundary conditions (8) we obtain eigenvalues equal to $k\pi/2l$ for the function T' and $k\pi/(R_0 - r_0)$ for the function T". Accordingly, the expressions for the functions F_i take the form

$$F_i(z) = \frac{a_{i0}}{2} + \sum_{k=1}^{\infty}\left(a_{ik}\cos\frac{k\pi z}{2l} + b_{ik}\sin\frac{k\pi z}{2l}\right)$$

$$(i = 0;\ 1),$$

$$F_i(r) = \frac{t_{i0}}{2} + \sum_{k=1}^{\infty}\left(t_{ik}\cos\frac{k\pi r}{R_0 - r_0} + s_{ik}\sin\frac{k\pi r}{R_0 - r_0}\right)$$

$$(i = 2;\ 3); \qquad (10)$$

$$a_{ik} = \frac{1}{l}\int_{-l}^{l} F_i(z)\cos\frac{k\pi z}{2l}\,dz;$$

$$b_{ik} = \frac{1}{l}\int_{-l}^{l} F_i(z)\sin\frac{k\pi z}{2l}\,dz \quad (i = 0;\ 1);$$

$$t_{ik} = \frac{2}{R_0 - r_0}\int_{r_0}^{R_0} F_i(r)\cos\frac{k\pi r}{R_0 - r_0}\,dr;$$

$$s_{ik} = \frac{2}{R_0 - r_0}\int_{r_0}^{R_0} F_i(r)\sin\frac{k\pi r}{R_0 - r_0}\,dr \quad (i = 2;\ 3). \qquad (11)$$

The general solution of the first equation of system (1) with boundary conditions (8) is obtained in the form

$$T = A_1 + A_2\ln\frac{r}{r_0} +$$

$$+ \sum_{k=1}^{\infty}\Big\{[B'_{1k}I_0(u_k r) + B'_{2k}K_0(u_k r)]\cos(u_k z) +$$

$$+ [B'_{3k}I_0(u_k r) + B'_{4k}K_0(u_k r)]\sin(u_k z) +$$

$$+ \left[B''_{1k}I_0\left(\frac{z + 2l}{v_k}\right) + B''_{2k}K_0\left(\frac{z + 2l}{v_k}\right)\right]\cos\frac{r}{v_k} +$$

$$+ \left[B''_{3k}I_0\left(\frac{z + 2l}{v_k}\right) + B''_{4k}K_0\left(\frac{z + 2l}{v_k}\right)\right]\sin\frac{r}{v_k}\Big\}. \qquad (12)$$

The constants of integration found from boundary conditions (8) are evaluated using the formulas

$$A_1 = \frac{cr_0 a_{00} + a_{10}}{2(cr_0\beta_b + \beta_0)}, \quad A_2 = \frac{a_{10} - 2A_1\beta_0}{2c},$$

$$B'_{1k} = D_{1k}(a_{0k}p_{4k} - a_{1k}p_{2k}), \quad B'_{2k} = D_{1k}(a_{1k}p_{1k} - a_{0k}p_{3k}),$$

$$B'_{3k} = D_{1k}(b_{0k}p_{4k} - b_{1k}p_{2k}), \quad B'_{4k} = D_{1k}(b_{1k}p_{1k} - b_{0k}p_{3k}),$$

$$B''_{1k} = D_{2k}(t_{2k}q_{4k} - t_{3k}q_{2k}), \quad B''_{2k} = D_{2k}(t_{3k}q_{1k} - t_{2k}q_{3k}),$$

$$B''_{3k} = D_{2k}(s_{2k}q_{4k} - s_{3k}q_{2k}), \quad B''_{4k} = D_{2k}(s_{3k}q_{1k} - s_{2k}q_{3k}), \qquad (13)$$

where

$$u_k = \frac{k\pi}{2l}; \quad v_k = \frac{R_0 - r_0}{k\pi}; \quad c = \frac{1}{R_0} + \beta_0\ln\frac{R_0}{r_0};$$

$$D_{1k} = (p_{1k}p_{4k} - p_{2k}p_{3k})^{-1}; \quad D_{2k} = (q_{1k}q_{4k} - q_{2k}q_{3k})^{-1};$$

$$p_{1k} = -u_k I_1(u_k r_0) + \beta_b I_0(u_k r_0);$$

$$p_{2k} = u_k K_1(u_k r_0) + \beta_b K_0(u_k r_0);$$

$$p_{3k} = u_k I_1(u_k R_0) + \beta_0 I_0(u_k R_0);$$

$$p_{4k} = -u_k K_1(u_k R_0) + \beta_0 K_0(u_k R_0);$$

$$q_{1k} = \frac{1}{v_k} I_1\left(\frac{3l}{v_k}\right) + \beta_1 I_0\left(\frac{3l}{v_k}\right);$$

$$q_{2k} = -\frac{1}{v_k} K_1\left(\frac{3l}{v_k}\right) + \beta_1 K_0\left(\frac{3l}{v_k}\right);$$

$$q_{3k} = \frac{1}{v_k} I_1\left(\frac{l}{v_k}\right) + \beta_1 I_0\left(\frac{l}{v_k}\right);$$

$$q_{4k} = -\frac{1}{v_k} K_1\left(\frac{l}{v_k}\right) + \beta_1 K_0\left(\frac{l}{v_k}\right).$$

At the points where the elements of the housing meet, we write the boundary conditions in the form

$$W_i = -2r_1\lambda\pi\delta\frac{dT_{ic}}{dr}; \quad r = r_1; \quad i = 1;\ 2;$$

$$T_{1c} = T_{0c}; \quad \frac{dT_{1c}}{dr} = \frac{dT_{0c}}{dz}; \quad r = R_1; \quad z = l_1;$$

$$T_{2c} = T_{0c}; \quad \frac{dT_{2c}}{dr} = \frac{dT_{0c}}{dz}; \quad r = R_1; \quad z = -l_1. \qquad (14)$$

The case $\gamma_1^2 > 0$, $\gamma_2^2 > 0$ corresponds to small values of δ and β_2. In this case we obtain the functions T_{ic} in the form

$$T_{0c} = C_{01}\sin(\gamma_1 z) + C_{02}\cos(\gamma_1 z) + \Theta;$$

$$T_{ic} = C_{i1}J_0(\gamma_2 r) + C_{i2}N_0(\gamma_2 r) + \Theta \quad (i = 1;\ 2). \qquad (15)$$

The constants of integration C_{ik} are computed from the formulas

$$C_{01} = \frac{\sigma_1 - 1}{2(\mu\eta\sigma_1^2 - \sigma_2\sigma_3)}\left[\varkappa_1(\eta\sigma_1 - \sigma_2) - \varkappa_2(\eta\sigma_1 + \sigma_2)\right],$$

$$C_{02} = \frac{\sigma_1 - 1}{2(\mu\eta\sigma_1^2 - \sigma_2\sigma_3)}\left[\varkappa_2(\mu\sigma_1 - \sigma_3) + \varkappa_1(\mu\sigma_1 + \sigma_3)\right],$$

$$C_{i2} = \frac{\eta C_{02} + (-1)^{i+1}\mu C_{01} - \varkappa_i}{\tau_2 - \tau_1}; \quad C_{i1} = \varkappa_i - \tau_1 C_{i2},$$

where we have introduced the notation

$$\tau_1 = \frac{N_1(\gamma_2 r_1)}{J_1(\gamma_2 r_1)}; \quad \tau_2 = \frac{N_0(\gamma_2 R_1)}{J_0(\gamma_2 R_1)}; \qquad (16)$$

$$\varkappa_i = \frac{W_i}{2r_1\pi\delta J_1(\gamma_2 r_1)\lambda};$$

$$\sigma_1 = \frac{1}{\tau_2 - \tau_1}\left[\frac{N_1(\gamma_2 R_1)}{J_1(\gamma_2 R_1)} - \tau_1\right];$$

$$\sigma_2 = -\frac{\gamma_1 \sin(\gamma_1 l_1)}{\gamma_2 J_1(\gamma_2 R_1)}; \quad \sigma_3 = -\frac{\gamma_1 \cos(\gamma_1 l_1)}{\gamma_2 J_1(\gamma_2 R_1)};$$

$$\mu = \frac{\sin(\gamma_1 l_1)}{J_0(\gamma_2 R_1)}; \quad \eta = \frac{\cos(\gamma_1 l_1)}{J_0(\gamma_2 R_1)}. \tag{17}$$

For the case $\gamma_1^2 < 0$, $\gamma_2^2 < 0$, i.e., for relatively large δ and β_2, we obtain

$$T_{0c} = C_{01}\exp(-\gamma_1 z) + C_{02}\exp(\gamma_1 z) + \Theta;$$

$$T_{ic} = C_{i1} I_0(\gamma_2 r) - C_{i2} K_0(\gamma_2 r) - \Theta \quad (i = 1;\ 2). \tag{18}$$

Then, instead of expressions (16) and (17), we have

$$C_{01} = \frac{(\sigma_1 + 1)[\varkappa_2(\mu\sigma_1 - \sigma_2) - \varkappa_1(\eta\sigma_1 - \sigma_3)]}{\sigma_1^2(\mu^2 + \eta^2) - \sigma_2^2 - \sigma_3^2},$$

$$C_{02} = \frac{(\sigma_1 + 1)[\varkappa_1(\mu\sigma_1 + \sigma_2) - \varkappa_2(\eta\sigma_1 + \sigma_3)]}{\sigma_1^2(\mu^2 + \eta^2) - \sigma_2^2 - \sigma_3^2},$$

$$C_{12} = \frac{\eta C_{01} + \mu C_{02} + \varkappa_1}{\tau_2 - \tau_1}, \quad C_{22} = \frac{\mu C_{01} + \eta C_{02} + \varkappa_2}{\tau_2 - \tau_1},$$

$$C_{i1} = \tau_1 C_{i2} - \varkappa_i, \tag{19}$$

the quantities introduced in (17) taking other values, namely,

$$\tau_1 = \frac{K_1(\gamma_2 r_1)}{I_1(\gamma_2 r_1)}; \quad \tau_2 = \frac{K_0(\gamma_2 R_1)}{I_0(\gamma_2 R_1)};$$

$$\varkappa_i = \frac{W_i}{2 r_1 \lambda\pi\delta I_1(\gamma_2 r_1)}; \quad \sigma_1 = \frac{1}{\tau_2 - \tau_1}\left[\frac{K_1(\gamma_2 R_1)}{I_1(\gamma_2 R_1)} - \tau_1\right];$$

$$\sigma_2 = -\frac{\gamma_1 \exp(\gamma_1 l_1)}{\gamma_2 I_1(\gamma_2 R_1)}; \quad \sigma_3 = -\frac{\gamma_1 \exp(-\gamma_1 l_1)}{\gamma_2 I_1(\gamma_2 R_1)};$$

$$\mu = \frac{\exp(\gamma_1 l_1)}{I_0(\gamma_2 R_1)}; \quad \eta = \frac{\exp(-\gamma_1 l_1)}{I_0(\gamma_2 R_1)}. \tag{20}$$

Earlier, in constructing Eqs. (1) it was assumed that the temperature gradients $\partial T_{0c}/\partial r$ and $\partial T_{ic}/\partial z$ were equal to the ratio of the temperature drop at the different surfaces of the housing to the thickness of the housing. Therefore, for the functions F_i given by Eqs. (9), after substituting the ratios for the derivatives, we obtain the relations

$$F_1 = \frac{\beta_0}{\delta\beta_0 - 1}(\delta\beta_0 T_{0c} - \Theta),$$

$$F_{i+1} = \frac{\beta_2}{\delta\beta_2 - 1}(\delta\beta_2 T_{ic} - \Theta) \quad (i = 1;\ 2),$$

which close the system of equations needed to calculate the temperature fields of the coaxial cylinders in question.

NOTATION

T_r is the temperature of rotor; T_{0c} is the temperature of cylindrical part of housing; T_{1c} and T_{2c} are the temperature of right and left ends of housing; β_b, β_0, and β_1 are the relative heat transfer coefficients for corresponding parts of the rotor boundary layer; β_2 is the relative heat transfer coefficient for lateral parts of housing boundary layer; β_3 is the relative heat transfer coefficient for outer surface of housing; T_{st} is the temperature distribution along surface of stator; Θ is the temperature of medium surrounding housing; W_1 and W_2 are the heat output of right and left bearings; A_i, B_i, and C_i are arbitrary constants of integration; F_i are functions, reduced to center of gap, determining the boundary conditions for the rotor; a, b, t, and s are the coefficients of Fourier expansions of the functions F_i; m_k are eigenvalues; $I_i(x)$ and $K_i(x)$ are modified Bessel functions of order i; λ is the thermal conductivity; r_0 is the inside radius of rotor; R_0 is the outside radius of rotor; $2l$ is the length of rotor; δ is the thickness of housing; r_1 is the radius of outer ring of bearing; $R_1 = R_0 + \delta$ and $l_1 = l + \delta$ are the characteristic dimensions of housing; r and z are cylindrical coordinates.

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2 July 1966

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